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1992 J. Phys. A: Math. Gen. 25 4189

(http://iopscience.iop.org/0305-4470/25/15/026)

J. Phys. A: Math. Gen. 25 (1992) 4189–4198. Printed in the UK

On the quantum mechanical propagator for driven coupled harmonic oscillators

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Received 25 June 1991, in final form 6 April 1992

Abstract. The exact propagator beyond and at caustics for a pair of coupled and driven oscillators with different frequencies and masses is calculated using the path-integral approach. The exact wavefunctions and energies are also presented. Finally the propagator is re-calculated through an alternative method, using the ζ-function.

1. Introduction

Recently some interest has been shown in the quantization of the coupled and driven harmonic oscillator [1–3]. In fact this potential can describe situations in many areas of physics such as superconducting quantum-interference devices [4], quantum non-demolition measurements [5] and magnetohydrodynamics [6].

The purpose of the present paper is to discuss the problem of two harmonic oscillators with different frequencies and masses, which are coupled through an arbitrary strength parameter. As far as we know, until now this problem has been considered only in the case of equal frequencies and masses [1–3], and for a particular strength of the coupling parameter [2].

This paper is organized as follow: in section 2 we calculate the propagator beyond and at caustics; in section 3 we obtain the normalized wavefunctions and energy spectrum; in section 4 we compare our results with previous ones by studying a particular case; in section 5 the propagator is calculated by an alternative method using the ζ-function approach. In section 6 we make our final considerations.

2. Propagator beyond and at caustics

The Lagrangian that will be treated here is written as

$$L=\sum_{j=1,2}(m_j/2)[\dot{x}_j^2-\omega_j^2x_j^2+2(f_j(t)/m_j)x_j]-\lambda x_1x_2 \tag{1}$$

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which represents a pair of coupled and driven anisotropic harmonic oscillators. In this section we will use Feynman's approach to quantization. The propagator is written as

$$K(x'', x'; \tau) = \int \mathscr{D}x_1 \mathscr{D}x_2 \exp\left[(i/\hbar)\int_{t'}^{t''} dt L(\dot{x}, x)\right] \tag{2}$$

where $x \equiv (x_1, x_2)$ and $\mathscr{D}x_1 \mathscr{D}x_2$ is the functional measure.

In order to quantize the above potential we will make some coordinate transformations so that we map the Lagrangian (1) into that of a pair of free particles. The decoupling transformation is given by

$$\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} (m/m_1)^{1/2}\cos\phi & (m/m_1)^{1/2}\sin\phi \\ -(m/m_2)^{1/2}\sin\phi & (m/m_2)^{1/2}\cos\phi \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \tag{3}$$

where m is an arbitrary parameter with dimension of mass. This leads us to the new Lagrangian

$$L' = (m/2)(\dot{y}_1^2 + \dot{y}_2^2) - \alpha y_1^2 - \beta y_2^2 - \gamma y_1 y_2 + F_1 y_1 + F_2 y_2 \tag{4}$$

where

$$\alpha = (m\omega_1^2/2)\cos^2\phi + (m\omega_2^2/2)\sin^2\phi - (\lambda m/2\sqrt{m_1 m_2})\sin(2\phi) \tag{5a}$$

$$\beta = (m\omega_1^2/2)\sin^2\phi + (m\omega_2^2/2)\cos^2\phi + (\lambda m/2\sqrt{m_1 m_2})\sin(2\phi) \tag{5b}$$

$$\gamma = (m/2)(\omega_1^2 - \omega_2^2)\sin(2\phi) + (\lambda m/\sqrt{m_1 m_2})\cos(2\phi) \tag{5c}$$

and

$$F_1 = \sqrt{m/m_1}\, f_1 \cos\phi - \sqrt{m/m_2}\, f_2 \sin\phi \tag{5d}$$

$$F_2 = \sqrt{m/m_1}\, f_1 \sin\phi + \sqrt{m/m_2}\, f_2 \cos\phi. \tag{5e}$$

Also, the path-integral measures changes ($\mathscr{D}x_1 \mathscr{D}x_2 = J\mathscr{D}y_1 \mathscr{D}y_2$) through a Jacobian that is given by: $J = \sqrt{m_1 m_2}/m$. In order to eliminate the coupling between y coordinates, we impose the condition $\gamma = 0$, and so

$$\tan(2\phi) = (2\lambda/\sqrt{m_1 m_2}(\omega_2^2 - \omega_1^2)). \tag{6}$$

Solving this to obtain the decoupling angle ϕ, we find two solutions which are equivalent and only interchange the role of the new coordinates y_1 and y_2, so that no physical difference appears between them. One of the solutions is

$$\cos\phi = [(1+R)/2]^{1/2} \tag{7a}$$

with

$$R = \frac{\sqrt{m_1 m_2 (\omega_2^2 - \omega_1^2)^2}}{\sqrt{4\lambda^2 + m_1 m_2 (\omega_2^2 - \omega_1^2)^2}}. \tag{7b}$$

With this solution we calculate α, β, F_1 and F_2, to obtain the Lagrangian

$$L = (m/2) \sum_{i=1,2} [\dot{y}_i^2 - \Omega_i^2 y_i^2 + (2/m) F_i y_i] \tag{8}$$

where

$$\Omega_1^2 = (1/2)[\omega_1^2 + \omega_2^2 - [(\omega_1^2 - \omega_2^2)^2 + 4\lambda^2/m_1 m_2]^{1/2}] \tag{9a}$$

$$\Omega_2^2 = (1/2)[\omega_1^2 + \omega_2^2 + [(\omega_1^2 - \omega_2^2)^2 + 4\lambda^2/m_1 m_2]^{1/2}] \tag{9b}$$

and

$$F_1=\sqrt{m/2m_1}\,f_1(1+R)^{1/2}-\sqrt{m/2m_2}\,f_2(1-R)^{1/2} \tag{9c}$$

$$F_2=\sqrt{m/2m_1}\,f_1(1-R)^{1/2}+\sqrt{m/2m_2}\,f_2(1+R)^{1/2}. \tag{9d}$$

At this point we have reduced the problem to that of two driven harmonic oscillators. We can, therefore, use the expression for the propagator for this system as given in Feynman and Hibbs [7]. For later convenience, we set $f(t'')\equiv f''$ and $f(t')\equiv f'$, for any function $f(t)$, where t' and t'' are the initial and final instants. Furthermore, we remember that the total propagator is written as

$$K(y'',y';\tau)=(\sqrt{m_1m_2}/m)K_1(y_1'',y_1';\tau)K_2(y_2'',y_2';\tau). \tag{10}$$

The desired propagator is obtained by returning to the original coordinates

$$\begin{vmatrix} y_1 \\ y_2 \end{vmatrix}=\begin{vmatrix} (m_1/m)^{1/2}\cos\phi & -(m_2/m)^{1/2}\sin\phi \\ (m_1/m)^{1/2}\sin\phi & (m_2/m)^{1/2}\cos\phi \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}+(1/\sqrt{m})\begin{vmatrix} \eta_1 \\ \eta_2 \end{vmatrix}. \tag{11}$$

Using the expressions for the propagator of a driven oscillator [7], substituting this in (10), and after lengthy but straightforward calculations, we can compute the propagator

$$\begin{aligned}
K(x'',x':\tau)={}&(1/2\pi\hbar \mathrm{i})\left[\frac{m_1m_2\Omega_1\Omega_2}{\sin(\Omega_1\tau)\sin(\Omega_2\tau)}\right]^{1/2}\\
&\times\exp\{-(\mathrm{i}/2\hbar)[\dot\eta_1(\eta_1+2\sqrt{m_1}\,Cx_1-2\sqrt{m_2}\,Sx_2)]_{t'}^{t''}\}\\
&\times\exp\left\{\frac{\mathrm{i}\Omega_1}{2\hbar\sin(\Omega_1\tau}[\cos(\Omega_1\tau)(m_1C^2x_1''^2+m_2S^2x_2''^2\right.\\
&-2\sqrt{m_1m_2}\,SCx_1''x_2''+\eta_1''^2+2\sqrt{m_1}\,C\eta_1''x_1''-2\sqrt{m_2}\,S\eta_1''x_2''\\
&+(x_1'',x_2'',\eta_1'')\to(x_1',x_2',\eta_1'))-2m_1C^2x_1'x_1''\\
&+2\sqrt{m_1m_2}\,SCx_1'x_2''-2\sqrt{m_1}\,C\eta_1''x_1'+2\sqrt{m_1m_2}\,SCx_2'x_1''\\
&-2m_2S^2x_2'x_2''+2\sqrt{m_2}\,S\eta_1'x_2''-2\sqrt{m_1}\,C\eta_1'x_1''\\
&\left.+2\sqrt{m_2}\,S\eta_1''x_2'-2\eta_1'\eta_1'']\right\}\\
&\times\exp\{\text{The above expressions with } 1\leftrightarrow 2 \text{ and } S\to -S\}\\
&\times\exp\left\{-(\mathrm{i}/2\hbar)\int_{t'}^{t''}\mathrm{d}\lambda\right.\\
&\left.\times[(f_1(\lambda)/\sqrt{m_1})(C+S)+(f_2(\lambda)/\sqrt{m_2})(C-S)]\right\}
\end{aligned} \tag{12}$$

where we define $S\equiv\sin\phi$, $C\equiv\cos\phi$ and

$$\begin{aligned}
\eta_i(t)={}&\left(\frac{1}{\sqrt{m}\sin(\Omega_i t)}\right)\left\{\int_{t'}^{t}\mathrm{d}\xi F_i(\xi)\sin(\Omega_i(\xi-t'))\sin(\Omega_i(t''-t))\right.\\
&\left.+\int_{t}^{t''}\mathrm{d}\xi F_i(\xi)\sin(\Omega_i(t-t'))\sin(\Omega_i(t''-\xi))\right\}
\end{aligned} \tag{13}$$

in order to simplify the expression of the propagator. However, as we can easily verify, this propagator diverges for certain time intervals, so, it is necessary to calculate it beyond caustics by applying the extended Feynman's formula [8, 9], obtaining

$$\begin{aligned}K_i(y_i'', y_i'; \tau) = &\left(\frac{m\Omega_i}{2\pi \mathrm{i}\hbar|\sin\sigma_i|}\right)^{1/2} \exp\left(\frac{\mathrm{i}}{2\hbar}[\dot{\eta}_i(\eta_i - 2\sqrt{m}\, y_i)]_{t'}^{t''}\right)\\ &\times \exp\left(\frac{\mathrm{i}m\Omega_i}{2\hbar\sin\sigma_i}[(y_i''^2 + y_i'^2)\cos\sigma_i - 2y_i'y_i'']\right)\\ &\times \exp\left(-\frac{\mathrm{i}}{2\hbar}\int_{t'}^{t''} \mathrm{d}\lambda\, \eta_i(\lambda)F_i(\lambda)/\sqrt{m}\right)\\ &\times \exp[-(\mathrm{i}\pi/2)\,\mathrm{Int}(\sigma_i/\pi)] \end{aligned} \tag{14}$$

with $\sigma_i = \Omega_i\tau$, and $\mathrm{Int}(\sigma_i/\pi)$ stands for the greatest integer which is less than or equal to σ_i/π. But the above propagator is only valid for time intervals between the successive caustics when $\sigma_i = k_i\pi (k_i \in \mathbb{Z})$. At caustics it is necessary to use a modified semigroup property of the propagator in order to calculate it [9]:

$$\begin{aligned}K_i(y_i'', y_i'; \sigma_i = k_i\pi) = &\exp(-\mathrm{i}k_i\pi/2)|K_i(y_i'', y_i; t''-t)|\\ &\times |K_i(y_i, y_i'; t-t')| \int_{-\infty}^{+\infty} \mathrm{d}y_i\\ &\times \exp\{(\mathrm{i}/\hbar)[S_{\mathrm{cl}}(y_i'', y_i; t''-t) + S_{\mathrm{cl}}(y_i, y_i'; t-t')]\}\end{aligned} \tag{15}$$

where $S_{\mathrm{cl}}(\cdot)$ is the classical action functional. Evaluating the integral in (15) with the help of equation (14), we obtain the propagator at caustics

$$\begin{aligned}K_i(y_i'', y_i'; \sigma_i = k_i\pi) = &\exp(-\mathrm{i}k_i\pi/2)\exp\{(\mathrm{i}/2\hbar)[\dot{\eta}_i(\eta_i - 2\sqrt{m}y_i)]_{t'}^{t''}\}\\ &\times \exp\left[-(\mathrm{i}/2\hbar)\int_{t'}^{t''} \mathrm{d}\lambda\, \eta_i(\lambda)F_i(\lambda)/\sqrt{m}\right]\\ &\times \delta[\sin(\Omega_i(t-t'))y_i'' - \sin(\Omega_i(t''-t))y_i'].\end{aligned} \tag{16}$$

Here we observe that the appearance of the Dirac delta function is understood, in terms of Feynman's method, from the existence of an infinite number of classical trajectories at caustics.

As the frequencies Ω_1 and Ω_2 (in general) are different then, for a given time interval, there are many different situations in which we will construct the total propagator; they are

$$K(x'', x'; \tau) = (\sqrt{m_1 m_2}/m)K_1(\sigma_1 \neq k_1\pi)K_2(\sigma_2 \neq k_2\pi) \tag{17a}$$

$$K(x'', x'; \tau) = (\sqrt{m_1 m_2}/m)K_1(\sigma_1 \neq k_1\pi)K_2(\sigma_2 = k_2\pi) \tag{17b}$$

$$K(x'', x'; \tau) = (\sqrt{m_1 m_2}/m)K_1(\sigma_1 = k_1\pi)K_2(\sigma_2 \neq k_2\pi) \tag{17c}$$

$$K(x'', x'; \tau) = (\sqrt{m_1 m_2}/m)K_l(\sigma_l = k_l\pi)K_2(\sigma_2 = k_2\pi) \tag{17d}$$

where $K_i(\sigma_i) \equiv K_i(x'', x'; \sigma_i)$. For each above possibility we can use the propagators (14) and (16) in order to obtain the total propagator.

3. Energy and wavefunctions

In this section we calculate the wavefunctions and the energy spectrum. For this we will make use of the expansion of the propagator in terms of the wavefunctions

$$K_i(y_i'', y_i'; \tau) = \sum_{n_i=0}^{\infty} \psi_{n_i}^*(y_i', t')\psi_{n_i}(y_i'', t'') \tag{18}$$

and with help of Mehler's formula [10]

$$\exp[-(u^2+v^2-2uvz)/(1-z^2)]$$
$$= (1-z^2)^{1/2}\exp[-(u^2+v^2)]\sum_{n=0}^{\infty}(z^n/n!)H_n(u)H_n(v) \tag{19}$$

with $u=(m\Omega_i t'/\hbar)^{1/2}y_i'$, $v=(m\Omega_i t''/\hbar)^{1/2}y_i''$, $z=\exp[-\mathrm{i}\hbar\Omega_i\tau]$ and $n=n_i$, we can rewrite (12) in the form of (18) and so obtain the wavefunctions

$$\begin{aligned}
\psi_{n_1n_2}(x_1, x_2, t)& \\
= \exp\{&-\mathrm{i}[(n_1+1/2)\Omega_1+(n_2+1/2)\Omega_2]t\}2^{-(n_1+n_2)/2} \\
&\times (n_1!\,n_2!)^{-1/2}\left[\frac{m_1m_2\Omega_1\Omega_2}{\pi^2\hbar^2}\right]^{1/4} \\
&\times \exp\{(1/2\hbar)[-\Omega_1(\sqrt{m_1}\,Cx_1-\sqrt{m_2}\,Sx_2+\eta_1)^2 \\
&-\Omega_2(\sqrt{m_1}\,Sx_1+\sqrt{m_2}\,Cx_2+\eta_2)^2 \\
&-\mathrm{i}\dot\eta_1[\eta_1-2(\sqrt{m_1}\,Cx_1-\sqrt{m_2}\,Sx_2) \\
&-\mathrm{i}\dot\eta_2[\eta_2-2(\sqrt{m_1}\,Sx_1+\sqrt{m_2}\,Cx_2)]]\} \\
&\times \exp\left\{-(\mathrm{i}/2\hbar)\int^t \mathrm{d}\lambda[(\eta_1/\sqrt{m_1})(f_1C-f_2S)+(\eta_2/\sqrt{m_2})(f_1S+f_2C)]\right\} \\
&\times H_{n_1}[(\Omega_1/\hbar)^{1/2}(\sqrt{m_1}\,Cx_1-\sqrt{m_2}\,Sx_2+\eta_1)] \\
&\times H_{n_2}[(\Omega_2/\hbar)^{1/2}(\sqrt{m_1}\,Sx_1+\sqrt{m_2}\,Cx_2+\eta_2)].
\end{aligned} \tag{20}$$

From the above expression it is easy to see that the energy spectrum for the non-driven case ($f_1=f_2=0$) is given by

$$E_{n_1,n_2} = (n_1+1/2)\hbar\Omega_1+(n_2+1/2)\hbar\Omega_2. \tag{21}$$

This energy spectrum shows us that there will exist degeneracies, provided that the frequencies Ω_1 and Ω_2 are related conveniently. In this case we will have $E_{n_1,n_2}=E_{n_1',n_2'}$ and so we come to

$$\Delta n_1\Omega_1+\Delta n_2\Omega_2=0 \qquad \text{or } \Delta n_1/\Delta n_2=\Omega_2/\Omega_1 \tag{22}$$

where $\Delta n_i \equiv n_i - n_i'$. As the n_i's are integers, then any time that the rate of the transformed frequencies is a rational fraction the system will have degeneracies. As the frequencies Ω_1 and Ω_2 are functions of all parameters in the Lagrangian we see that, when certain relations between the parameters hold, the system will be degenerate.

4. Particular cases

Now we calculate the particular case in which the two oscillators have the same mass and frequency and are non-driven. After this we obtain the case studied by Yeon *et al* [2], in order to verify our calculations in an explicit and known example.

In the particular case we have the Lagrangian

$$L = m(\dot{x}_1^2 + \dot{x}_2^2)/2 - m\omega^2(x_1^2 + x_2^2)/2 - \lambda x_1 x_2. \tag{23}$$

Substituting the above parameters in the expression for the frequencies Ω_1 and Ω_2 given in equation (9), we obtain

$$\Omega_1^2 = \omega^2 - \lambda/m \qquad \Omega_2^2 = \omega^2 + \lambda/m \tag{24}$$

and $\cos\phi = 1/\sqrt{2} = \sin\phi$, that corresponds to a rotation of $\pi/4$ about the coordinates origin. In this case the propagator beyond caustics reads

$$\begin{aligned} K(x'', x'; \tau) = {} & (m/2\pi\hbar \mathrm{i}) \left[\frac{\Omega_1\Omega_2}{|\sin(\Omega_1\tau)||\sin(\Omega_2\tau)|}\right]^{1/2} \exp\left\{\frac{\mathrm{i}m\Omega_1}{4\hbar\sin(\Omega_1\tau)}\right. \\ & \times [\cos(\Omega_1\tau)(x_1''^2 + x_2''^2 + x_1'^2 + x_2'^2 - 2(x_1''x_2'' + x_1'x_2')) \\ & \left. - 2(x_1'x_1'' + x_2'x_2'' - x_1'x_2'' - x_2'x_1'')]\right\} \\ & \times \exp\left\{\frac{\mathrm{i}m\Omega_2}{4\hbar\sin(\Omega_2\tau)}[\cos(\Omega_2\tau)(x_1''^2 + x_2''^2 + x_1'^2 + x_2'^2 + 2(x_1''x_2'' + x_1'x_2'))\right. \\ & \left. - 2(x_1'x_1'' + x_2'x_2'' + x_1'x_2'' + x_2'x_1'')]\right\} \\ & \times \exp\{-(\mathrm{i}\pi/2)[\mathrm{Int}(\Omega_1\tau/\pi) + \mathrm{Int}(\Omega_2\tau/\pi)]\}. \end{aligned} \tag{25}$$

with Ω_1 and Ω_2 given in equation (24). The propagator at caustics is written as

$$\begin{aligned} K(x'', x'; \tau) = {} & \sqrt{2}\exp[-\mathrm{i}\pi(k_1 + k_2)/2] \\ & \times \delta[\sin(\Omega_1(t - t'))(x_1'' - x_2'') - \sin(\Omega_1(t'' - t))(x_1' - x_2')] \\ & \times \delta[\sin(\Omega_2(t - t'))(x_1'' + x_2'') - \sin(\Omega_2(t'' - t))(x_1' + x_2')] \end{aligned} \tag{26}$$

and the wavefunctions are

$$\begin{aligned} \psi_{n_1 n_2}(x_1, x_2, t) & \\ = {} & \exp(-\mathrm{i}E_{n_1,n_2}t/\hbar)2^{-(n_1+n_2)/2}(n_1!\,n_2!)^{-1/2}\left[\frac{m^2\Omega_1\Omega_2}{\pi^2\hbar^2}\right]^{1/4} \\ & \times \exp\{-(m/4\hbar)[\Omega_1(x_1 - x_2)^2 + \Omega_2(x_1 + x_2)^2]\} \\ & \times H_{n_1}[(m\Omega_1/2\hbar)^{1/2}(x_1 - x_2)]H_{n_2}[(m\Omega_2/2\hbar)^{1/2}(x_1 + x_2)]. \end{aligned} \tag{27}$$

Furthermore, in the case treated by Yeon *et al* [2], the Lagrangian that corresponds to their Hamiltonian (2.1) with vanishing driven forces, is given in our case by

$$L = m(\dot{x}_1^2 + \dot{x}_2^2)/2 - m\omega^2(x_1^2 + x_2^2) + m\omega^2 x_1 x_2 \tag{28}$$

from which we can see that it is equal to our Lagrangian (23) only if we make the identifications $\omega^2 \to 2\omega^2$ and $\lambda \to -m\omega^2$. Taking this care we recall the energy spectrum of the cited paper, because the frequencies (24) become $\Omega_1^2 = \omega^2$ and $\Omega_2^2 = 3\omega^2$, so

$$E_{n_1,n_2} = [(n_1 + 1/2) + \sqrt{3}\,(n_2 + 1/2)]\hbar\omega \tag{29}$$

as obtained in equation (6.10) of [2].

5. ζ-function approach

Up to now our solutions have utilized path-integral methods. Now we will perform an alternative calculation of the propagator (12). In this alternative approach use is made of a ζ-function to calculate the determinant for a given differential operator. This is an interesting technique that was recently applied in similar types of problems [11, 12], and that has been used extensively in quantum field theories [13].

In order to use such an approach we shall do the usual expansion of the trajectory [7] about the classical configuration

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\mathrm{cl}}(t) + \boldsymbol{\eta}(t) \tag{30}$$

where $\boldsymbol{x}_{\mathrm{cl}}(t)$ is the classical solution and $\boldsymbol{\eta}(t)$ is the quantum fluctuation that obeys the Dirichlet boundary conditions

$$\boldsymbol{\eta}(0) = 0 = \boldsymbol{\eta}(\tau) \tag{31}$$

with τ being the time-interval between the initial and final points of the trajectory.

Substituting (31) in the expression (2) and using the classical equations of motion, we obtain

$$K(\boldsymbol{x}'', \boldsymbol{x}'; \tau) = F(\tau)\exp((\mathrm{i}/\hbar)S_{\mathrm{cl}}) \tag{32}$$

where S_{cl} is the classical action. Which can be obtained by using the classical solution of the harmonic oscillators with frequencies Ω_1 and Ω_2 calculated as above and then returning to the original coordinates.

Furthermore it is not difficult to see that the pre-exponential factor is written as

$$F(\tau) = \int \mathscr{D}\eta_1 \mathscr{D}\eta_2 \exp\left\{(\mathrm{i}/2\hbar)\int_0^\tau \mathrm{d}t[m_1\dot\eta_1^2 + m_2\dot\eta_2^2 - m_1\omega_1^2\eta_1^2 - m_2\omega_2^2\eta_2^2 - 2\lambda\eta_1\eta_2]\right\}. \tag{33a}$$

However it is more convenient to deal with convergent integrals rather than oscillatory ones. This can be achieved by making a continuation to an imaginary time, followed by a Wick rotation. So we have

$$\beta = \mathrm{i}t \qquad \mathrm{d}/\mathrm{d}t = \mathrm{i}\,\mathrm{d}/\mathrm{d}\beta \qquad \Sigma = \mathrm{i}\tau$$

which can be substituted in expression ($33a$) for the pre-exponential factor. So, after straightforward calculations we find

$$F(\Sigma) = \int \mathscr{D}\eta \exp\left\{-(1/2\hbar)\int \mathrm{d}\beta\,\eta^T(\beta)\mathbb{D}\eta(\beta)\right\} \tag{33b}$$

with the defined quantities

$$\eta \equiv \begin{vmatrix}\eta_1\\ \eta_2\end{vmatrix} \qquad \eta^T \equiv |\eta_1\eta_2| \tag{34a}$$

and

$$\mathbb{D} \equiv \begin{vmatrix} m_1(-\mathrm{d}^2/\mathrm{d}\beta^2+\omega_1^2) & \lambda \\ \lambda & m_2(-\mathrm{d}/\mathrm{d}\beta^2+\omega_2^2) \end{vmatrix}. \tag{34b}$$

Now we can diagonalize the operator $\mathbb{D}$ by using the transformation matrices in (3) (denoted $\mathbb{M}$). For this we make the transformation

$$\eta = (\hbar/m)^{1/2}\mathbb{M}\rho \tag{35}$$

so that

$$F(\Sigma) = J(\hbar/m)\int \mathscr{D}\rho \exp\left[-(1/2)\int_0^\tau \mathrm{d}\beta\rho^T\mathbb{D}'\rho\right] \tag{36}$$

with

$$\mathbb{D}' \equiv \mathbb{M}^T\mathbb{D}\mathbb{M}. \tag{37}$$

This diagonalized operator is given by

$$\mathbb{D}' \equiv \begin{vmatrix} -\mathrm{d}^2/\mathrm{d}\beta^2+\Omega_1^2 & 0 \\ 0 & -\mathrm{d}/\mathrm{d}\beta^2+\Omega_2^2 \end{vmatrix} \tag{38}$$

where Ω_1^2 and Ω_2^2 are given in equation (9) and the pre-exponential factor is identified as being proportional to the determinant of the operator $\mathbb{D}'$,

$$F(\Sigma) = \sqrt{m_1 m_2}\, C|\det \mathbb{D}'|^{-1} \tag{39}$$

C is an arbitrary constant to be determined and is associated with the phase of the propagator. Considering the determinant of the operator $\mathbb{D}'$ as the product of their eigenvalues, it is easy to see that it diverges. Some regularization must to be done in order to regularize this quantity. The regularization scheme that we choose here utilizes the ζ-function.

The generalized ζ-function for a given operator is defined as

$$\zeta(s) \equiv \sum_n \lambda_n^{-s} \tag{40}$$

and after necessary analytical continuation [14], it can be shown that [11, 12]

$$\det \mathbb{D}' = \exp[-(\mathrm{d}\zeta/\mathrm{d}s)|_{s=0}]. \tag{41}$$

On the other hand, the eigenvalues of the operator $\mathbb{D}'$ are given by

$$\varepsilon_{n_1} = (n_1\pi/\Sigma)^2 + \Omega_1^2 \text{ and } \varepsilon_{n_2} = (n_2\pi/\Sigma)^2 + \Omega_2^2 \tag{42}$$

so that we can obtain the determinant from

$$\det \mathbb{D}' = \prod_{n_1=1}^{\infty} \varepsilon_{n_1} \prod_{n_2=1}^{\infty} \varepsilon_{n_2} \tag{43}$$

and each product can be obtained from a ζ-function calculation. However, for convenience, we decompose the eigenvalue ε_{n_j} $(j=1,2)$, so that

$$\begin{aligned} \det \mathbb{D}' &= \prod_{n_1=1}^{\infty} \varepsilon_{n_1}^{+}\varepsilon_{n_1}^{-} \prod_{n_2=1}^{\infty} \varepsilon_{n_2}^{+}\varepsilon_{n_2}^{-} \\ &= \det \mathbb{D}_1'^{+} \det \mathbb{D}_1'^{-} \det \mathbb{D}_2'^{+} \det \mathbb{D}_2'^{-} \end{aligned} \tag{44a}$$

with

$$\varepsilon_{n_j}^{\pm} = (n_j\pi/\Sigma) \pm \mathrm{i}\Omega_j \tag{44b}$$

and, using the generalized Riemann ζ-function [12], defined as

$$\zeta_{\mathrm{R}}(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s} \qquad \mathrm{Re}(s) > 1. \tag{45}$$

From equation (44a) we see that we must calculate four determinants. For a given $\varepsilon_{n_j}^{\pm}$ the generalized ζ-function is given by

$$\begin{aligned}\zeta_{\mathrm{R},n_j}^{\pm}(s) &= (\pi/\Sigma)^{-s} \sum_{n_j=0}^{\infty} (n_j \pm \mathrm{i}(\Sigma/\pi)\Omega_j)^{-s} \mp (\mathrm{i}\Omega_j)^{-s} \\ &= (\pi/\Sigma)^{-s} \zeta_{\mathrm{R},n_j}(s, \pm\mathrm{i}(\Sigma/\pi)\Omega_j) \mp (\mathrm{i}\Omega_j)^{-s}\end{aligned} \tag{46}$$

which has been manipulated in order to use the generalized ζ-function whose sum begins in $n_j = 0$. Using now the property of the generalized ζ-function [15]

$$\begin{aligned}&\zeta_{\mathrm{R}}(s=0, a) = 1/2 - a \\ &\mathrm{d}\zeta_{\mathrm{R}}(s, a)/\mathrm{d}s|_{s=0} = \ln\Gamma(a) - (1/2)\ln(2\pi)\end{aligned} \tag{47}$$

and after some straightforward calculations we obtain

$$\det \mathbb{D}_{n_j}^{\prime\pm} = \frac{\sqrt{2\pi}}{(\pm\mathrm{i}\Omega_j)\Gamma(\pm\mathrm{i}\Sigma\Omega_j/\pi)} \left(\frac{\pi}{\Sigma}\right)^{(1/2 \pm \mathrm{i}\Sigma\Omega_j/\pi)} \tag{48}$$

which, after the use of the identity,

$$\Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin(\pi x)} \tag{49}$$

the substitution of (48) in (44a) and the use of the initial value condition for the propagator

$$\lim_{\tau\to 0} K(x'', x'; \tau) = \delta(x'' - x') \tag{50}$$

gives the pre-exponential factor

$$F(\tau) = (2\pi\mathrm{i}\hbar)^{-1} \left[\frac{m_1 m_2 \Omega_1 \Omega_2}{\sin(\Omega_1\tau)\sin(\Omega_2\tau)}\right]^{1/2} \tag{51}$$

where we have used that $\Sigma = \mathrm{i}\tau$. We can see that it is really the correct pre-exponential factor as calculated above, using the Feynman approach, in equation (12).

6. Conclusions

In this work we have solved, both by a path-integral approach and by a ζ-function approach, the quantum mechanical problem of two coupled harmonic oscillators with different masses and frequencies. So, we have enlarged the list of quantum mechanical systems that can be exactly solved through these methods. Using Feynman's method, we have also calculated the propagator beyond and at caustics.

The wavefunctions in general, and the eigenvalues in the particular case of a non-driven system, were calculated and compared with previous calculations.

A remarkable feature is the appearance of degeneracies in the energy spectrum, provided that certain relations between the potential parameters hold. This suggests the existence of some hidden symmetry in the system, in analogy with the accidental degeneracy appearing in a two-dimensional isotropic harmonic oscillator.

Acknowledgments

The author thanks CNPq (Conselho Nacional de Desenvolvimento Científico e Technológico) Brazil for partial financial support.

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